DIAS-STP-90-22

Functional Integrals in the Strong Coupling Regime and the Polaron Self-energy

G.V. Efimov, G. Ganbold

ABSTRACT:

A general method is presented for representation of functional integrals in the strong coupling regime.

This method is applied to the computing of polaron ground energy for large coupling constant.

1. INTRODUCTION

Feyman's path-integral or the functional integral is widely used in quantum mechanics, statistical physics, quantum field theory and so on. Solution of many physical and mathematical problems can be represented in the form of functional integrals. The only method which can be considered as mathematically well established consists in perturbation calculations. The standard form of these integrals is shown in (2.1) where the Gaussian measure with an appropriate Green function describes the noninteracting free system and a nonlinear part describes the interaction in this system. The practical purpose is to compute this integral. However, there are great difficulties in the performing of these calculations. In quantum field theory functional integrals are defined as mathematical objects in a perturbation sense only. There are no reliable general methods if we want to get out of the perturbation method and calculate functional integrals in the strong coupling regime. Therefore different variational approaches are widely used for this aim [1,2]. The variational method has got a good reputation in describing such physical values as a ground state energy due to its low sensibility to errors in the choice of a trial wave function. The main defect of variational calculations is that we do not know how close the obtained variational estimation to the true described value and we can not compute the next correlation in order to improve the main variational estimation.

The aim of this paper is to formulate a general method of calculation of functional integral in the strong coupling regime. Our idea is the following. We propose that the functional integral is of the Gaussian type in the strong coupling regime but with an other Green function in the Gaussian measure. The contribution to self-energy which are propotional to the tadpole Feynman diagrams are the main one to the formation of the new state. Thus the mathematical problem is to take into account them correctly. It can be done by introducing the concept of the normal product according to the given Gaussian measure. We formulate the equations which make possible to perform this program. As a result we obtain the equivalent representation of the initial functional integral, in which the main contributions of the strong interaction are concentrated in the new Green function defining the Gaussian measure and in an explicit expression for the ground state energy. This representation permits us to compute small perturbation corrections.

We put this method to the problem of poloron ground state energy. The poloron is one of the simplest nontrivial models standing between quantum field theory and quantum mechanics [1]. There are many papers devoted to this problem [3-15]. In this paper we obtain the poloron ground energy in the strong coupling regime. The comparison of our result with others is done in table 1.

2. GENERAL FORMALISM

In this section we formulate our method of calculation of functional integrals defined on the Gaussian measure. We shall consider the functional integrals of the following general type

$$Z_\Gamma(g) = N_0 \int \delta\varphi \exp\left\{-\frac{1}{2}(\varphi D_0^{-1} \varphi) + g\, W[\varphi]\right\}. \qquad (2.1)$$

Here we have introduced the following notations.

$$(\varphi D_0^{-1} \varphi) = \int_\Gamma dx \int_\Gamma dy\, \varphi(x) D_0^{-1}(x,y)\, \varphi(y). \qquad (2.2)$$

The integration in (2.2) is performed over a region $\Gamma \subseteq \mathbb{R}^d$ (d= 1, 2, ...). Usually the region is a box

$$\Gamma = \{ x: \; a_j \leq x_j \leq b_j \; , j = 1, \ldots, d \} . \tag{2.3}$$

D_o^{-1} (x,y) is a differential operator defined on functions $\varphi(x)$ with appropriate boundary conditions. For example

$$D_o^{-1}(x,y) = \left(-\frac{\partial^2}{\partial x^2} + m_o^2 \right) \delta(x-y) \tag{2.4}$$

with periodic boundary conditions. The Green function D_o (x,y) satisfies

$$\int_\Gamma dy \, D_o^{-1}(x_1, y) \, D_o(y, x_2) = \delta(x_1 - x_2) . \tag{2.5}$$

The normalization constant N_o in (2.1) is defined by the condition

$$N_o \int \delta\varphi \exp\{ -\tfrac{1}{2} (\varphi D_o^{-1} \varphi) \} = 1 \tag{2.6}$$

and

$$N_o = \frac{1}{\sqrt{\det D_o}} = \sqrt{\det D_o^{-1}}$$

The interaction functional $W[\varphi]$ can be written in a general form

$$W[\varphi] = \int d\mu_a \, e^{i(a\varphi)} \tag{2.7}$$

where

$$(a\varphi) = \int_\Gamma dx \, a(x) \varphi(x)$$

and $d\mu_a$ is a measure. For example,

$$W[\varphi] = \int_\Gamma dx \, U(\varphi(x)) = \int_\Gamma dx \int \frac{dk}{2\pi} \tilde{U}(k) \exp\left\{ i \int_\Gamma dy \, k \, \varphi(y) \delta(x-y) \right\} \tag{2.8}$$

The parameter g is a coupling constant.

We consider that the integral (2.1) does exist as functional integral and can be calculated by the perturbation method for a small coupling constant g. Our aim is to give a representation of our integral in the strong coupling regime. In other words we want to obtain a representation in which all main contributions of strong interaction are concentrated in the Green function of the Gaussian measure.

Let us perform the following transformations in the integral (2.1):

$$\begin{aligned} &\varphi(x) \to \varphi(x) + f(x), \\ &D_0^{-1}(x,y) \to D^{-1}(x,y) \end{aligned} \tag{2.9}$$

where f(x) and D^{-1} (x,y) are arbitrary functions. The Green function D (x,y) satisfies

$$\int_\Gamma dy\, D^{-1}(x_1,y)\, D(y,x_2) = \delta(x_1-x_2)\,.$$

Then the functional integral becomes the form

$$\begin{aligned} Z_\Gamma(g) = \exp\left\{\frac{1}{2}\ln\det\frac{D}{D_0} - \frac{1}{2}(f D_0^{-1} f)\right\} \cdot \\ \cdot \int d\sigma_D \exp\left\{ W_{int}[\varphi, f, D]\right\} \end{aligned} \tag{2.10}$$

Here

$$d\sigma_D = N\delta\varphi \exp\left\{-\frac{1}{2}(\varphi D^{-1}\varphi)\right\}, \quad N = \sqrt{\det D^{-1}},$$

$$W_{int} = g\, W[\varphi+f] - (f D_0^{-1}\varphi) - \frac{1}{2}\left(\varphi[D_0^{-1} - D^{-1}]\varphi\right). \tag{2.11}$$

Now let us introduce the concept of the normal product according to the given Gaussian measure $d\sigma_D$. It means that

$$e^{i(a\varphi)} = :e^{i(a\varphi)}: \, e^{-\frac{1}{2}(aDa)} \tag{2.12}$$

so that

$$\int d\sigma_D :e^{i(a\varphi)}: = 1$$

or

$$\int d\sigma_D :\varphi(x_1)\cdot\ldots\cdot\varphi(x_n): = 0 .$$

Then the interaction functional W_{int} (2.11) can be rewritten

$$\begin{aligned} W_{int} = {} & g\int d\mu_a e^{i(a\varphi)-\frac{1}{2}(aDa)} :e^{i(a\varphi)} - 1 - i(a\varphi) + \frac{1}{2}(a\varphi)^2: + \\ & + \left[g\int d\mu_a e^{i(a\varphi)-\frac{1}{2}(aDa)} - \frac{1}{2}\left([D_0^{-1} - D^{-1}]D\right)\right] + \\ & + \left[g\int d\mu_a e^{i(a\varphi)-\frac{1}{2}(aDa)} i(a\varphi) - (b D_0^{-1}\varphi)\right] - \\ & - \frac{1}{2} :\left[g\int d\mu_a e^{i(a\varphi)-\frac{1}{2}(aDa)} (a\varphi)^2 + (\varphi[D_0^{-1} - D^{-1}]\varphi)\right]: . \end{aligned} \tag{2.13}$$

Our basic idea is that the main contribution into functional integral is concentrated in the Gaussian measure. It means that the lenear and quadratic terms over the integration variables $\varphi(x)$ should be absent in the interaction functional W_{int} (2.13). Thus, we obtain two equations

$$\begin{aligned} & g\int d\mu_a \, i a(x) e^{i(ab)-\frac{1}{2}(aDa)} - \int dy D_0^{-1}(x,y) b(y) = 0 \\ & g\int d\mu_a \, a(x)a(y) e^{i(ab)-\frac{1}{2}(aDa)} + D_0^{-1}(x,y) - D^{-1}(x,y) = 0. \end{aligned} \tag{2.14}$$

These equations provide the removing oof these terms.

Let us introduce the functional

$$\overline{W}[b] = \int d\mu_a \exp\left\{ i(ab) - \frac{1}{2}(aDa)\right\} . \tag{2.15}$$

Then the equations (2.14) can be written in the form

$$b(x) = g \int_\Gamma dy\, D_0(x,y) \frac{\delta}{\delta b(y)} \overline{W}[b],$$
$$D(x_1,x_2) = D_0(x_1,x_2) + \iint_\Gamma dy_1 dy_2\, D_0(x_1,y_1) \frac{\delta^2 \overline{W}[b]}{\delta b(y_1)\delta b(y_2)} D(y_2,x_2). \tag{2.16}$$

These equations define the functions b(x) and D(x,y).

Finally we obtain

$$Z_\Gamma(g) = \exp\{W_0\} \int d\sigma_D \exp\{g\, \overline{W}_2[\varphi]\}. \tag{2.17}$$

Here

$$W_0 = \frac{1}{2} \ln \det \frac{D}{D_0} - \frac{1}{2}(b D_0^{-1} b) - \frac{1}{2}([D_0^{-1} - D^{-1}]D) + g\, \overline{W}[b],$$
$$\overline{W}_2[\varphi] = g \int d\mu_a\, e^{i(ab) - \frac{1}{2}(aDa)} :e^{i(a\varphi)} - 1 - i(a\varphi) + \frac{1}{2}(a\varphi)^2: . \tag{2.18}$$

The functions b(x) and D(x,y) are defined by the equations (2.16)

One should stress that the representations (2.1) and (2.17) are equivalent. Therefore our mathematical object $Z_\Gamma(g)$ has at least two different representations (2.1) and (2.17). In principal we can get other representations if the equations (2.16) have different solutions. We shall choose the representation in which the perturbation corrections connected with $g\,\overline{W}_2$ or $g\,W$ are at a minimum for the given parameters in the interaction functional $g\,W$.

All our transformations and equations (2.16) are valid for real and complex functions in the functional integral (2.1).

In the case of real functional integrals the equality (2.19) and equations (2.16) lead to the following conclusion. The interaction functional $\overline{W}_2[\varphi]$ in (2.17) satisfies

$$\int d\sigma_D\, \overline{W}_2[\varphi] = 0. \tag{2.19}$$

Using the Jensen's inequality we have

$$Z_\Gamma(g) \geqslant \exp\{ W_0 \} \tag{2.20}$$

so that W_0 defines the lower estimation for our functional integral.

On the other hand, one can check easily that the equations (2.16) define the minimum of the functional W_0 in (2.18). Thus the inequality (2.20) is a variation estimation of our functional integral (2.1). Moreover, the representation (2.17) permits us to calculate the perturbation corrections to W_0 developing the functional integral (2.17) over $g\, W_2[\varphi]$.

3. GROUND STATE ENERGY OF POLORON

The ground state energy $E(\alpha)$ of the Fröhlich polorons is defined by the expression

$$Z_\beta(\alpha) = N_0 \int\limits_{\vec{r}(0)=\vec{r}(\beta)} \delta\vec{r}\, \exp\left\{ -\frac{1}{2}\int_0^\beta ds\, \dot{\vec{r}}^{\,2}(s) + \frac{\alpha}{\sqrt{8}} \iint_0^\beta \frac{ds\,dt\, e^{-|s-t|}}{|\vec{r}(t)-\vec{r}(s)|} \right\}, \tag{3.1}$$

$$E(\alpha) = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln Z_\beta(\alpha). \tag{3.2}$$

Our aim is to get the behaviour of the function $E(\alpha)$ for $\alpha \to \infty$. We apply the method described in the previous section to the functional integral (3.1). This integral (3.1) can be written in the form (2.1):

$$Z_T(\alpha) = N_0 \int\limits_{\vec{r}(-T)=\vec{r}(T)} \delta\vec{r}\, \exp\left\{ -\frac{1}{2}\iint_{-T}^{T} ds\,dt\, \vec{r}(t) D_0^{-1}(t,s) \vec{r}(s) + \alpha U[\vec{r}] \right\} \tag{3.3}$$

$$T = \frac{1}{2}\beta \quad , \quad D_0^{-1}(t,s) = -\frac{\partial^2}{\partial t^2}\delta(t-s), \tag{3.4}$$

$$\alpha U[\vec{r}] = \frac{\alpha}{\sqrt{8}} \iint_{-T}^{T} ds\,dt\, e^{-|s-t|} \int \frac{d\vec{k}}{2\pi^2 \vec{k}^2}\, e^{i\vec{k}(\vec{r}(s)-\vec{r}(t))}, \tag{3.5}$$

$$E(\alpha) = -\lim_{T\to\infty} \frac{1}{2T} \ln Z_T(\alpha).$$

The Green function $D_0(t,s)$ satisfying the periodic boundary conditions is
$$D_0(t,s) = -\frac{1}{2}|t-s| - \frac{ts}{T} \xrightarrow[T\to\infty]{} D_0(t-s) = -\frac{1}{2}|t-s|,$$

$$\tilde{D}_0(k) = \int dx\, e^{ikx} D_0(x) = \frac{1}{2}\left[\frac{1}{(k+i0)^2} + \frac{1}{(k-i0)^2}\right]. \tag{3.6}$$

We want to find the limit in (3.5). Therefore we shall consider that $T = \frac{1}{2}\beta$ is asymptotically large. In this limit the Green functions $D_0(t,s)$ and $D(t,s)$ are translationally invariant, i.e.

$$D_0(t,s) = D_0(t-s), \quad D(t,s) = D(t-s).$$

The interaction function $\overline{U}[\vec{b}]$ (2.15) can be written:

$$\alpha \overline{U}[\vec{b}] = \frac{\alpha}{\sqrt{8}} \iint_{-T}^{T} ds\, dt\, e^{-|t-s|} \int \frac{d\vec{u}}{2\pi^2 \vec{u}^2}\, e^{i\vec{u}(\vec{b}(s) - \vec{b}(t)) - \vec{u}^2 F(t-s)}, \tag{3.7}$$

$$F(t-s) = D(0) - D(t-s).$$

Now let us consider the equations (2.16). We choose $\vec{b}(s) = 0$ as solution of the first equation (2.16) because this solution seems to be natural as it follows from the explicit form of the integral (3.1). Then we obtain

$$\Sigma_{ij}(s_1 - s_2) = \delta_{ij}\Sigma(s_1 - s_2) = \left.\frac{\delta^2 \overline{U}[b]}{\delta b_i(s_1)\delta b_j(s_2)}\right|_{b=0} = \tag{3.8}$$
$$= \delta_{ij}\frac{1}{6\sqrt{2\pi}} \cdot \left[\delta(s_1-s_2)\int_{-\infty}^{\infty} \frac{dt\, e^{-|t|}}{F^{3/2}(t)} - \frac{e^{-|s_1-s_2|}}{F^{3/2}(s_1-s_2)}\right]$$

and

$$\tilde{\Sigma}(k) = \int_{-\infty}^{\infty} ds\, e^{iks}\Sigma(s) = \frac{1}{3\sqrt{2\pi}} \int_0^{\infty} \frac{dt\, e^{-t}(1-\cos kt)}{F^{3/2}(t)}. \tag{3.9}$$

The second equation in (2.16) has the following form

$$\tilde{D}(k) = \tilde{D}_0(k) - \tilde{D}_0(k)\,\alpha\tilde{\Sigma}(k)\,\tilde{D}(k)$$

and
$$\widetilde{D}(\kappa) = \frac{1}{\widetilde{D}_0^{-1}(\kappa) + \alpha \widetilde{\Sigma}(\kappa)} . \tag{3.10}$$

Finally we obtain for the function $F(s)$ the following equation
$$F(s) = \int_0^\infty \frac{d\kappa}{\pi} \cdot \frac{1 - \cos \kappa s}{\kappa^2 + \alpha \widetilde{\Sigma}(\kappa)} \tag{3.11}$$
where $\widetilde{\Sigma}(\kappa)$ is defined by (3.9).

The energy $E_0(\alpha)$ which is defined by W_0 in the representation (2.18) is
$$E_0(\alpha) = -\frac{3}{2\pi} \int_0^\infty d\kappa \left[\ln \kappa^2 \widetilde{D}(\kappa) - \kappa^2 \widetilde{D}(\kappa) + 1 \right] - \frac{\alpha}{\sqrt{2\pi}} \int_0^\infty \frac{dt\, e^{-t}}{\sqrt{F(t)}} . \tag{3.12}$$

The representation (2.18) for the polaron functional integral looks as ($T = \frac{1}{2}\beta$ asymptotically large)
$$Z_T(\alpha) = e^{-2T E_0(\alpha)} \cdot J_T(\alpha), \tag{3.13}$$
$$J_T(\alpha) = N \int\limits_{\vec{r}(-T) = \vec{r}(T)} \delta\vec{r}\, \exp\left\{ -\frac{1}{2} \iint_{-T}^{T} dt\, ds\, \vec{r}(t) D^{-1}(t-s) \vec{r}(s) + \alpha \overline{U}_2[\vec{r}] \right\} \tag{3.14}$$
where
$$\alpha \overline{U}_2[\vec{r}] = \frac{\alpha}{\sqrt{8}} \iint_{-T}^{T} ds\, dt\, e^{-|t-s|} \int \frac{d\vec{u}}{2\pi^2 \vec{u}^2} e^{-\vec{u}^2 F(t-s)} \times$$
$$\times : e^{i\vec{u}(\vec{r}(t) - \vec{r}(s))} - 1 + \frac{1}{6} \vec{u}^2 \cdot (\vec{r}(t) - \vec{r}(s))^2 : . \tag{3.15}$$
The functions $D(t)$ and $F(t)$ are defined by the equations (3.7) and (3.11).

It should be stressed that the representation (3.15) is completly equivalent to the initial representation (3.1) for asymptotically large $T=\frac{1}{2}\beta$.

4. THE STRONG COUPLING LIMIT.

In this section we obtain the representation for the ground state energy of polaron for asymptotically large α. Let us consider the functional integral (3.14). In the formulas (3.14) and (3.15) we introduce new variables

$$\vec{K}=\sqrt{\mu}\,\vec{p}\,,\quad t=\frac{u}{\mu}\,,\quad s=\frac{v}{\mu}\,,\quad \vec{r}(t)=\frac{\vec{\rho}(u)}{\sqrt{\mu}} \tag{4.1}$$

where μ is a parameter depending on α. Then all our formulas become the form

$$J_\beta(\alpha)=N\int\delta\vec{\rho}\,\exp\Big\{-\frac{1}{2}\iint_{-T\mu}^{T\mu}du\,dv\,\vec{\rho}(u)\frac{1}{\mu^3}D^{-1}\Big(\frac{u-v}{\mu}\Big)\vec{\rho}(v)+ \tag{4.2}$$
$$+\frac{\alpha}{\sqrt{8}}\iint_{-T\mu}^{T\mu}\frac{du\,dv}{\mu^{3/2}}e^{-\frac{|u-v|}{\mu}}\int\frac{d\vec{p}}{2\pi^2\vec{p}^2}e^{-\vec{p}^2\mu F\left(\frac{u-v}{\mu}\right)}:e^{i\vec{p}(\vec{\rho}(u)-\vec{\rho}(v))}-1+\frac{1}{6}\vec{p}^2(\vec{\rho}(u)-\vec{\rho}(v))^2:\Big\}$$

Now let us consider the equation (3.11) where we introduce

$$\mu F\Big(\frac{u}{\mu}\Big)=\mu^2\int_0^\infty\frac{dp}{\pi}\cdot\frac{1-\cos pu}{\mu^2p^2+\alpha\tilde{\Sigma}(\mu p)}\,,$$
$$\tilde{\Sigma}(\mu p)=\frac{1}{3\sqrt{2\pi}}\int_0^\infty dt\,e^{-t}F^{-\frac{3}{2}}(t)(1-\cos\mu p t). \tag{4.3}$$

Our basic assumption, which will be justified is that the parameter μ increases when $\alpha\to\infty$.

One can see that the equations (4.3) have the limit when $\mu\to\infty$ if

$$F(t)=\frac{1}{\mu}\phi(\mu t) \tag{4.4}$$

Then we have

$$\phi(u) = \int_0^\infty \frac{dp}{\pi} \cdot \frac{1 - \cos pu}{p^2 + \frac{\alpha}{\mu^2} \tilde{\Sigma}(\mu p)}, \tag{4.5}$$

$$\tilde{\Sigma}(\mu p) = \frac{\mu^{3/2}}{3\sqrt{2\pi}} \int_0^\infty \frac{dt\, e^{-t}(1-\cos \mu p t)}{\phi^{3/2}(\mu t)} \xrightarrow[\mu\to\infty]{} \frac{\mu^{3/2}}{3\sqrt{2\pi}\ \phi^{3/2}(\infty)}$$

and

$$\phi(u) = \int_0^\infty \frac{dp(1-\cos pu)}{\pi\,(p^2+1)} = \frac{1}{2}\left(1 - e^{-|u|}\right). \tag{4.6}$$

where the parameter μ is chosen is such a way that

$$\frac{\alpha}{\mu^2} \cdot \frac{\mu^{3/2}}{3\sqrt{2\pi}} \cdot \frac{1}{\phi^{3/2}(\infty)} = 1$$

The representation (4.6) leads to

$$\phi(\infty) = \frac{1}{2}$$

and

$$\mu = \frac{4\alpha^2}{9\pi}. \tag{4.7}$$

Thus we get, for our functions for

$$F(t) = \frac{1}{2\mu}\left(1 - e^{-\mu|t|}\right), \quad D(t) = \frac{1}{2\mu} e^{-\mu|t|} = \int \frac{dp}{2\pi} \frac{e^{ipt}}{p^2+\mu^2},$$

$$D^{-1}(t_1 - t_2) = \left(-\frac{\partial^2}{\partial t^2} + \mu^2\right)\delta(t_1 - t_2), \tag{4.8}$$

$$\frac{1}{\mu^3} D^{-1}\left(\frac{u-v}{\mu}\right) = \left(-\frac{\partial^2}{\partial u^2} + 1\right)\delta(u-v).$$

Substituting these formulas into (4.2) we obtain

$$J_\beta(\alpha) = N\int \delta\vec{\rho}\, \exp\left\{-\frac{1}{2}\int_{-T\mu}^{T\mu} du\left[\dot{\vec{\rho}}^{\,2}(u) + \vec{\rho}^{\,2}\right] + \alpha \overline{U}_2[\vec{\rho}]\right\} \tag{4.9}$$

where
$$\alpha \overline{U}_2[\vec{\rho}] = \frac{\alpha}{\sqrt{8\mu}} \iint_{-T\mu}^{T\mu} \frac{du\,dv}{\mu} e^{-\frac{|u-v|}{\mu}} \int \frac{d\vec{p}}{2\pi^2 \vec{p}^{\,2}} e^{-\vec{p}^{\,2}\phi(u-v)} \times$$
$$\times :\sum_{n=3}^{\infty} \frac{i^n}{n!}\Big[2(\vec{p}\vec{\rho}(u))^n + \sum_{m=1}^{n-1}(-)^m \frac{n!}{m!(n-m)!}(\vec{p}\vec{\rho}(u))^m(\vec{p}\vec{\rho}(v))^{n-m}\Big]: \tag{4.10}$$

In order to find the limit $\mu\to\infty$ in (4.10) let us introduce the new variable
$$v = \mu t + u$$
we get
$$\alpha \overline{U}_2[\vec{\rho}] = \sqrt{\frac{9\pi}{32}} \iint_S du\,dt\, e^{-|t|} \int \frac{d\vec{p}}{2\pi^2 \vec{p}^{\,2}} e^{-\vec{p}^{\,2}\phi(\mu t)} \times$$
$$\times :\sum_{n=3}^{\infty} \frac{i^n}{n!}\Big[2(\vec{p}\vec{\rho}(u))^n + \sum_{m=1}^{n-1}(-)^m \frac{n!}{m!(n-m)!}(\vec{p}\vec{\rho}(u))^m(\vec{p}\vec{\rho}(u+\mu t))^{n-m}\Big]: \tag{4.11}$$

The region S is shown on Fig.1. The second term in the square brackets in (4.11) disappears when $\mu\to\infty$ because this term leads in the higest perturbation orders to the products of convolutions of the type
$$\sim \prod_j e^{-|u-u_j|} e^{-|\mu t+u-u_j|} .$$

Thus for $\mu\to\infty$ we obtain
$$\alpha \overline{U}_2[\vec{\rho}] = \overline{W}[\vec{\rho}] = 3\sqrt{\frac{\pi}{2}} \int_{-T\mu}^{T\mu} du \int \frac{d\vec{p}}{2\pi^2 \vec{p}^{\,2}} e^{-\frac{\vec{p}^{\,2}}{2}} \times$$
$$\times : e^{i\vec{p}\vec{\rho}(u)} - 1 + \frac{1}{2}(\vec{p}\vec{\rho}(u))^2 : . \tag{4.12}$$

It should be stressed that the normal ordering in (4.12) is taken according to the Gaussian measure in (4.9).

Thus the formulas (4.9) and (4.12) allow us to write the final representation

$$I_\beta = N \int\limits_{\vec{\rho}(0)=\vec{\rho}(\beta)} \delta\vec{\rho}\, \exp\Big\{-\frac{1}{2}\int_0^\beta du\,[\dot{\vec{\rho}}^2(u)+\vec{\rho}^2(u)] + \tag{4.13}$$
$$+3\sqrt{\frac{\pi}{2}}\int_0^\beta du \int \frac{d\vec{p}}{2\pi^2\vec{p}^2}\, e^{-\frac{\vec{p}^2}{2}} : e^{i\vec{p}\vec{\rho}(u)} - 1 + \frac{(\vec{p}\vec{\rho}(u))^2}{2} : \Big\}.$$

Here we have introduced β instead $2T\mu$ in (4.9) and (4.12). The formula (4.13) can be represented for the asymptotically large β in the form

$$I_\beta = \exp\Big\{\frac{1}{2}\iint_0^\beta du\,dv\, D(u-v)\frac{\delta^2}{\delta\rho_i(u)\delta\rho_i(v)}\Big\} \times \tag{4.14}$$
$$\times \exp\Big\{3\sqrt{\frac{\pi}{2}}\int_0^\beta du \int \frac{d\vec{p}}{2\pi^2\vec{p}^2}\, e^{-\frac{\vec{p}^2}{2}} : e^{i\vec{p}\vec{\rho}(u)} - 1 + \frac{1}{2}(\vec{p}\vec{\rho}(u))^2 :$$

where
$$D(u) = \frac{1}{2}e^{-|u|}.$$

Now let us go back to the ground state energy of polaron for asymptotic large α. This energy is defined by the formula (3.2) and according to the representations (3.13) and (4.13) it is equal

$$E = E_0 + \mu\,\mathcal{E} \tag{4.15}$$

where E_0 is defined by (3.12) and

$$\mathcal{E} = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln I_\beta . \tag{4.16}$$

The formula (3.12) gives for the functions $\tilde{D}(k^2)$ and $F(t)$ defined in (4.8)

$$E_0 = -\frac{\alpha^2}{3\pi}$$

and using the definition (4.7) one can get

$$E = -\frac{\alpha^2}{3\pi}\Big[1 + \frac{4}{3}\mathcal{E}\Big] . \tag{4.17}$$

Let us go to the representation (4.13) and let us come back to the ordinary operator product in the interaction function, one can obtain after simple transformations

$$I_\beta = N\int \delta\vec{\rho}\, \exp\left\{-\int_0^\beta dt\left[\frac{1}{2}\dot{\vec{\rho}}^{\,2}(t) - 3\sqrt{2}\int_0^1 ds\, e^{-\vec{\rho}^{\,2}(t)s^2} + \frac{15}{4}\right]\right\}. \quad (4.18)$$

It means that our parameter in (4.16) and (4.17) is the lowest proper value of the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\vec{\rho}^{\,2}} - 3\sqrt{2}\int_0^1 ds\, e^{-\vec{\rho}^{\,2}s^2} + \frac{15}{4} \quad (4.19)$$

Thus, we have reduced our problem to the solving of the Schrodinger equation

$$\left(-\frac{1}{2}\Delta - 3\sqrt{2}\int_0^1 ds\, e^{-\vec{\rho}^{\,2}s^2}\right)\Psi = \left(\mathcal{E} - \frac{15}{4}\right)\Psi. \quad (4.20)$$

We have calculated the parameter $\mathcal{E}$ (4.15) in the second and third perturbation orders using the representation (4.13). Our result is the following

$$E = -\frac{\alpha^2}{3\pi}\left[1+\gamma_2+\gamma_3\right] = -\frac{\alpha^2}{3\pi}1.01745\ldots = -\alpha^2 0.1080\ldots$$

$$\gamma_2 = -\frac{4}{3}\mathcal{E}_2 = 0.01566\ldots \quad (4.21)$$

$$\gamma_3 = -\frac{4}{3}\mathcal{E}_3 = 0.00178\ldots$$

The comparison of the results is shown is table 1.

ACKNOWLEDGEMENTS

One of the authors, G. Efimov, would like to thank Professor John Lewis for his hospitality at the Institute for Advance Studies, Dublin and the fruitfull discussions.

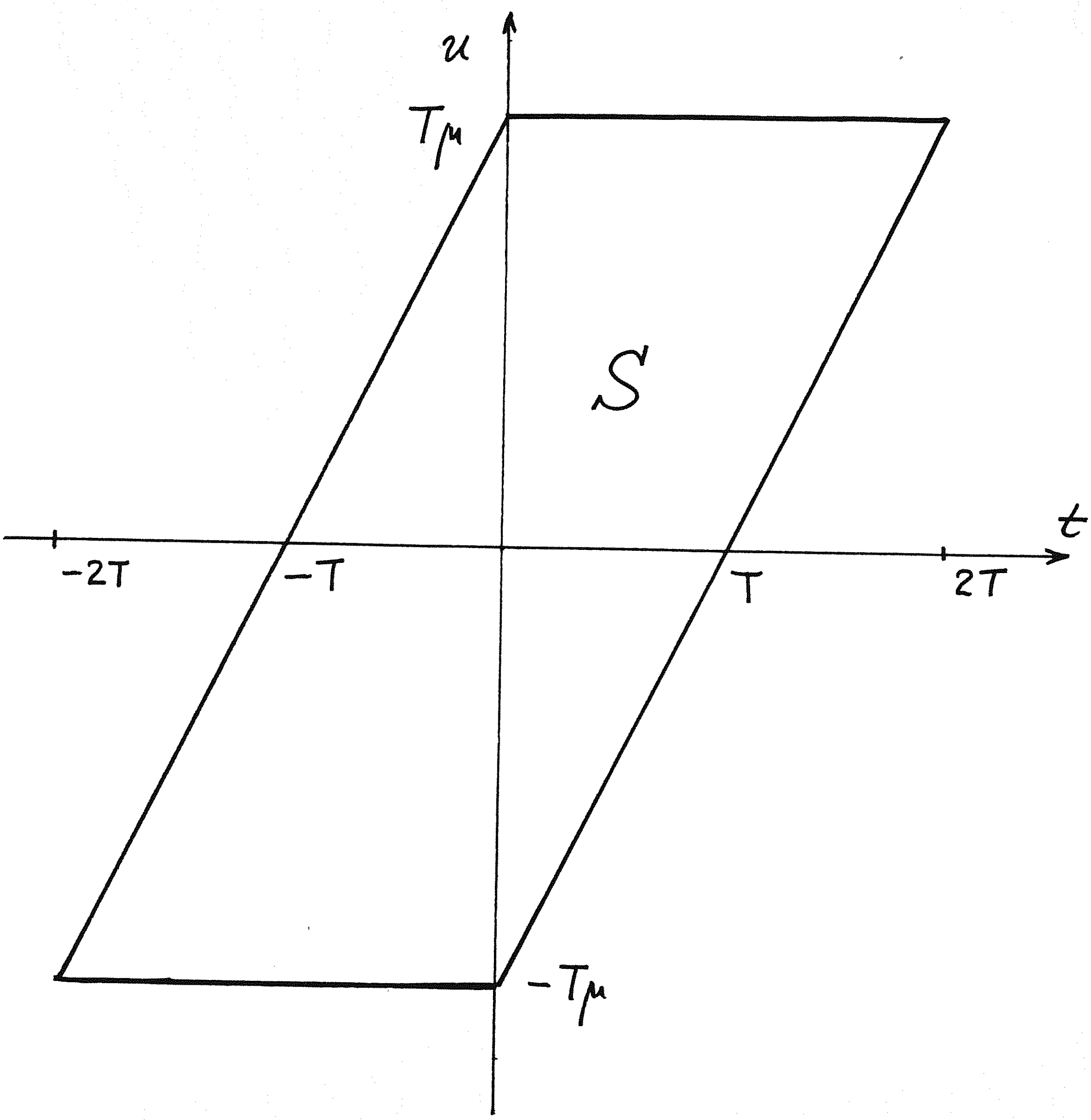

Fig. 1. The region of integration in the integral (4.11)

TABLE 1 The comparison of the results in the case of strong coupling (the coefficients of α^2)

AUTHORS		$\gamma/3\pi$
Feynman, Schultz	[15]	0.1061
Pekar	[7]	0.1088
Miyake (γ_{exact})	[8]	0.1085
Luttinger, Lu	[9]	0.1066
Tokuda	[10]	0.1061
Marshall, Mills	[11]	0.1078
Sheng, Dow	[12]	0.1065
Smondyrev	[14]	0.1092
Selyugin, Smondyrev	[15]	0.1085
Ours		0.1080

REFERENCES

1. R.P. Feynman, A.R. Hibbs. "Quantum Mechanics and Path Integrals" (McGraw-Hill Book Company, New York, 1965)
2. Proceedings of the international Workshop on "Variational calculations in Quantum Field Theory", edited by L. Polley, D.E.L. Pottinger (World Scientific, 1987)
3. M. Horst, V. Merkt and J.P. Kottaus, Phys.Rev.Lett., 50, 754 (1983)
4. S.A. Jackson and P.M. Platzman, phys. Rev., B24, 499 (1981)
5. T.D. Schultz, Phys.Rev., 116, 526 (1959)
6. R. Abe and K. Okamoto, Jour.Phys.Soc.Jap., 31, 1337 (1971)
7. S. Pekar, Zh.Eks.Theor.Fiz., 19, 796 (1954)
8. S. Miyake, Jour.Phys.Soc.Jap., 38, 181 (1975)
9. J.M. Lutinger and C-Y Lu, Phys.Rev., B21, 4251 (1982)
10. N. Tokuda, J. PHys., C13, L851 (1980)
11. J.T. Marshall and L.R. Mills, Phys.Rev., B2, 3143 (1970)
12. P. Sheng and J.D. Dow, Phys.Rev., B4, 1343 (1971)
13. F.M. Peeters, W. Xiaoguang and J.T. Devreese, Phys.Re., B33, 3926 (1986)
14. M.A. Smondyrev, preprint JINR P17-89-250 (1989)
15. O.V. Selyugin and M.A. Smondyrev, Phys.Stat.Sol.(b), 155, 155(1989)